

Relations

Group I

For each of the following sets of properties, give an example of a relation that satisfies them.

1. Not reflexive, not symmetric, and not transitive
2. Reflexive, not symmetric, and not transitive
3. Not reflexive, symmetric, and not transitive
4. Not reflexive, not symmetric, and transitive
5. Not reflexive, symmetric and transitive
6. Reflexive, not symmetric, and transitive
7. Reflexive, symmetric, and not transitive
8. Reflexive, symmetric, and transitive

Group II

Prove that the following relations are equivalence relations.

9. The relation A defined on \mathbb{R} as follows: $\forall x, y \in \mathbb{R}, x A y \leftrightarrow |x| = |y|$.
10. The congruence modulo 2 relation B defined on \mathbb{Z} as follows: $\forall x, y \in \mathbb{Z}, x B y \leftrightarrow x - y$ is even.
11. The relation C defined on the set S of strings on the alphabet $\{0,1\}$ defined as follows: $\forall x, y \in S, x C y \leftrightarrow x$ and y have the same number of ones.
12. The relation D defined on $\mathbb{R} \times \mathbb{R}$ as follows: $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}, (x_1, y_1) D (x_2, y_2) \leftrightarrow x_1 = x_2$.
13. The relation E defined on $\mathbb{Z} \times \mathbb{Z}$ as follows: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}, (a, b) E (c, d) \leftrightarrow a + d = c + b$.
14. The relation F defined on $\mathbb{Z} \times \mathbb{Z}$ as follows: $\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}, (a, b) F (c, d) \leftrightarrow ad = cb$.

Group III

Let R be an equivalence relation on a set A. Prove each of the statements directly from the definitions of equivalence relation and equivalence class.

15. $\forall a, b \in A$, if $a R b$, then $[a] = [b]$.
16. $\forall a, b \in A$, either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.
17. $\forall a \in A, a \in [a]$.
18. $\forall a, b \in A$, if $b \in [a]$, then $a R b$.
19. $\forall a, b, c \in A$, if $b R c$ and $c \in [a]$, then $b \in [a]$.
20. $\forall a, b \in A$, if $[a] = [b]$, then $a R b$.
21. $\forall a, b, x \in A$, if $a R b$ and $x \in [a]$, then $x \in [b]$.
22. $\forall a, b \in A$, if $a \in [b]$, then $[a] = [b]$.

Group IV

Prove that the following relations are partial order relations.

23. The relation G defined on $\mathbb{R} \times \mathbb{R}$ as follows: $\forall (a, b), (c, d) \in \mathbb{R}, (a, b) G (c, d) \leftrightarrow a \leq c$ and $b \leq d$.
24. The relation H defined on $\mathbb{R} \times \mathbb{R}$ as follows: $\forall (a, b), (c, d) \in \mathbb{R}, (a, b) H (c, d) \leftrightarrow a < c$ or $a = c$ and $b \leq d$.
25. Given a set M, the relation S defined on $\mathcal{P}(M)$ as follows: $\forall K, L \in \mathcal{P}(M), K S L \leftrightarrow K \subseteq L$.

Group V

For each of the following posets, give a Hasse diagram.

26. A poset that has at least 8 elements, exactly two of which are maximal elements and exactly two of which are minimal.
27. A poset that has at least 8 elements, a greatest element, and no least element.
28. A poset that has at least 8 elements, no greatest element, and a least element.
29. A poset that has at least 8 elements, a greatest element, and a least element.
30. A poset that has at least 8 elements and every chain has length 2.
31. A poset that has at least 8 elements and where there is a set of four elements that are incomparable.