

Function Proofs

Prove each of the following using logic, set properties, definitions, and/or with theorems earlier on this sheet.

Assume all sets are a subset of the universe U .

Group I

For each of the theorems in this group, let X and Y be sets. Let $f: X \rightarrow Y$ be a function. Let A and B be subsets of X . Let C and D be subsets of Y .

1. If $A \subseteq B$, then $f(A) \subseteq f(B)$.
2. $f(A \cap B) \subseteq f(A) \cap f(B)$.
3. $f(A \cup B) = f(A) \cup f(B)$.
4. $f(A) - f(B) \subseteq f(A - B)$.
5. If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.
6. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
7. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
8. $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Group II

For each of the theorems in this group, let X and Y be sets. Let $f: X \rightarrow Y$ be a function.

9. $i_Y \circ f = f$ and $f \circ i_X = f$.
10. $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.
11. For all subsets A of X , $A \subseteq f^{-1}(f(A))$.
12. For all subsets C of Y , $f(f^{-1}(C)) \subseteq C$.

Group III

For each of the theorems in this group, let X , Y , and Z be sets. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be functions.

13. If f and g are both one-to-one, then $g \circ f$ is one-to-one.
14. If f and g are both onto, then $g \circ f$ is onto.
15. If f and g are both bijections, then $g \circ f$ is a bijection.
16. If $g \circ f$ is one-to-one, then f is one-to-one.
17. If $g \circ f$ is onto, then g is onto.
18. If $g \circ f$ is a bijection, then f is one-to-one and g is onto.

Group IV

For each of the theorems in this group, let X and Y be sets. Let $f: X \rightarrow Y$ be a function.

19. The function f is onto if, and only if $f(X) = Y$.
20. The function f is onto if, and only if, for all subsets C of Y , $f(f^{-1}(C)) = C$.
21. The function f is one-to-one if, and only if, for all subsets A and B of X , $f(A \cap B) = f(A) \cap f(B)$.
22. The function f is one-to-one if, and only if, for all subsets A and B of X , $f(A - B) = f(A) - f(B)$.
23. The function f is one-to-one if, and only if, for all subsets A of X , $f^{-1}(f(A)) = A$.

Group V

24. Let X , Y , Z , and W be sets. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Z \rightarrow W$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.
25. Let X , Y , and Z be sets. Let $f: X \rightarrow Y$, $g: Y \rightarrow Z$, and $h: Y \rightarrow Z$ be functions. If $g \circ f = h \circ f$ and f is onto, then $g = h$.
26. Let X , Y , and Z be sets. Let $f: X \rightarrow Y$, $g: X \rightarrow Y$, and $h: Y \rightarrow Z$ be functions. If $h \circ f = h \circ g$ and h is one-to-one, then $f = g$.
27. Let X , Y , and Z be sets. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be bijections. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.